Approximating derivatives using interpolating polynomials

1. Approximate the derivative of te^{-t} at t = 2.5 and h = 0.1 and h = 0.05 using each of the formulas shown in class:

$$y^{(1)}(t) \approx \frac{y(t) - y(t-h)}{h}$$

$$y^{(1)}(t) \approx \frac{3y(t) - 4y(t-h) + y(t-2h)}{2h}$$

$$y^{(1)}(t) \approx \frac{y(t+h) - y(t-h)}{2h}$$

Answer:	–0.1251059133484300	–0.1232976066833525	–0.1230589226866095
	–0.1241358072742140	–0.1231657011999980	–0.1231103862051720

2. The derivative at t = 2.5 for the expression in Question 1 is approximately –0.1231274979358482. Indicate which formulas are O(h) and which are O(h^2), and support your claims based on the results shown in Question 1.

Answer: The errors, to four significant digits, of the above six results are

0.001978	0.0001701	–0.00006858
0.001008	0.00003820	–0.00001711

You will note the first error drops by approximately one half, while the next two drop by approximately one quarter.

3. An accurate sensor is reading a location at a rate of once every five seconds, and the reading is in meters travelled. The readings are as follows:

0, 0.06, 1.01, 4.89, 14.60, 33.10, 62.75, 104.71, 158.66, 222.86, 294.59

What is a reasonable integer approximation of the speed at the last reading in km/h?

Answer: The speed in m/s is $\frac{3 \cdot 294.59 - 4 \cdot 222.86 + 158.66}{2 \cdot 5} = 15.099$ and 1 m/s = 3.6 km/h, so a reasonable approximation of the speed is 54 km/h.

4. Prove that the formulas in Question 1 are O(h), O(h^2) and O(h^2), respectively.

Answer: See the course notes.

5. Suppose we use the approximation $y^{(1)}(t) \approx \frac{y(t-h) - y(t-2h)}{h}$ to approximate the derivative one time step into the future of the two samples at $t - h$ and $t - 2h$. What is the error of this approximation?

Answer: We have

$$y(t-h) = y(t) - y^{(1)}(t)h + \frac{1}{2}y^{(2)}(\tau_{-1})h^2$$
$$y(t-2h) = y(t) - 2y^{(1)}(t)h + 2y^{(2)}(\tau_{-2})h^2$$

Subtracting the second from the first, we have

$$y(t-h) - y(t-2h) = y^{(1)}(t)h + \left(\frac{1}{2}y^{(2)}(\tau_{-1}) - 2y^{(2)}(\tau_{-2})\right)h^2$$

We may therefore solve this as follows:

$$y^{(1)}(t) = \frac{y(t-h) - y(t-2h)}{h} - \left(\frac{1}{2}y^{(2)}(\tau_{-1}) - 2y^{(2)}(\tau_{-2})\right)h$$

The second expression is not a convex combination (with all coefficients greater than or equal to zero), so the best we can say is:

$$y^{(1)}(t) = \frac{y(t-h) - y(t-2h)}{h} + \frac{3}{2}y^{(2)}(t)h + O(h^2)$$

Thus, the error is $\frac{3}{2}y^{(2)}(t)h + O(h^2)$, which is O($h$), but with a much larger coefficient.

6. Suppose we use the approximation $y^{(1)}(t) \approx \frac{3y(t-h)-4y(t-2h)+y(t-3h)}{2h}$ to approximate the derivative one time step into the future of the three samples at $t-h$, $t-2h$ and $t-3h$. What is the error of this approximation?

Answer: Let's start with first-order Taylor series expansions:

$$\begin{aligned} y(t-h) &= y(t) - y^{(1)}(t)h + \frac{1}{2}y^{(2)}(\tau_{-1})h^2 \\ y(t-2h) &= y(t) - 2y^{(1)}(t)h + 2y^{(2)}(\tau_{-2})h^2 \\ y(t-3h) &= y(t) - 3y^{(1)}(t)h + \frac{9}{2}y^{(2)}(\tau_{-3})h^2 \end{aligned}$$

If, when we add these together, the errors all cancel out, then we would go back and use a 2nd-order Taylor series expansion.

Taking an appropriate linear combination of these, we have

$$3y(t-h)-4y(t-2h)+y(t-3h) = 2y^{(1)}(t)h + \left(\frac{3}{2}y^{(2)}(\tau_{-1}) - 8y^{(2)}(\tau_{-2}) + \frac{9}{2}y^{(2)}(\tau_{-3})\right)h^2$$

We may therefore solve this as follows:

$$y^{(1)}(t) = \frac{3y(t-h)-4y(t-2h)+y(t-3h)}{2h} - \frac{1}{2}\left(\frac{3}{2}y^{(2)}(\tau_{-1}) - 8y^{(2)}(\tau_{-2}) + \frac{9}{2}y^{(2)}(\tau_{-3})\right)h$$

The second expression is not a convex combination (with all coefficients greater than or equal to zero), so the best we can say is:

$$y^{(1)}(t) = \frac{3y(t-h)-4y(t-2h)+y(t-3h)}{2h} + y^{(2)}(t)h + O(h^2)$$

Thus, the error is $y^{(2)}(t)h + O(h^2)$, which is O(h) and not O(h^2).

7. A colleague suggests that a better approximation of the derivative for sampled data would be $y^{(1)}(t) \approx \frac{1.5y(t) - 0.5y(t-h)}{1.5h}$ as this puts the most emphasis on the more recent, and therefore most relevant, reading. What is the error of this formula?

Answer: From a 1st-order Taylor series, we have:

$$y(t-h) = y(t) - y^{(1)}(t)h + \frac{1}{2}y^{(2)}(\tau)h^2.$$

Thus, we have

$$1.5y(t-h) = 1.5y(t) - 1.5y^{(1)}(t)h + 0.75y^{(2)}(\tau)h^2$$

Solving this for the derivative, we have:

$$y^{(1)}(t) = \frac{1.5y(t) - 1.5y(t-h)}{1.5h} - 0.5y^{(2)}(\tau)h.$$

But the formula only uses $1.5y(t) - 0.5y(t-h)$, so part of the formula is incorporated into the error:

$$y^{(1)}(t) = \frac{1.5y(t) - 0.5y(t-h)}{1.5h} - \frac{y(t-h)}{1.5h} - 0.5y^{(2)}(\tau)h.$$

We note the error is $-\frac{y(t-h)}{1.5h} - 0.5y^{(2)}(\tau)h$, so this is actually O(1/h), meaning the smaller h is, the worse the approximation. Thus, this is a valid formula, but also one that has a result that is useless, except, perhaps, if we were approximating the derivative of a function that was itself essentially equal to zero.

8. Approximate the second derivative of te^{-t} at t = 2.5 and h = 0.1 and h = 0.05 using each of the formulas shown in class:

$$y^{(2)}(t) \approx \frac{y(t+h) - 2y(t) + y(t-h)}{h^2}$$

$$y^{(2)}(t) \approx \frac{y(t) - 2y(t-h) + y(t-2h)}{h^2}$$

$$y^{(2)}(t) \approx \frac{2y(t) - 5y(t-h) + 4y(t-2h) - y(t-3h)}{h^2}$$

Answer:	0.04093981323641	0.03616613330155	0.04239674992034
	0.04101684276168	0.03880424296864	0.04135154960976

9. The second derivative at t = 2.5 for the expression in Question 8 is approximately 0.0410424993119494. Indicate which formulas are O(h) and which are O(h^2), and support your claims based on the results shown in Question 8.

Answer: The errors, to four significant digits, of the above six results are

0.0001027	0.004876	–0.001354
0.00002566	0.002238	–0.0003091

You will note the first and third errors drop by approximately one quarter, while the second drops by approximately one half.

10. An accurate sensor is reading a location at a rate of once every five seconds, and the reading is in meters travelled. The readings are as follows:

0, 0.06, 1.01, 4.89, 14.60, 33.10, 62.75, 104.71, 158.66, 222.86, 294.59

What is a reasonable integer approximation of the acceleration at the last reading in km/h/(10s)? That is, what is the change in speed in km/h over a period of 10 seconds.

Answer: The speed in m/s^2 is $\frac{2 \cdot 294.59 - 5 \cdot 222.86 + 4 \cdot 158.66 - 104.71}{5^2} = 0.1924$ and we have that 1 m/s^2 = 36 km/h/(10s), so a reasonable approximation of the acceleration is 7 km/h/(10s).

11. Prove that the formulas in Question 8 are O(h^2), O(h) and O(h^2), respectively.

Answer: See the course notes. The last requires you to have three 4$^{\text{th}}$-order Taylor series expansions.

12. In class, we found the two approximations of the 2nd derivative by finding an interpolating quadratic and taking the second derivative thereof. Suppose, however, you recall that

$$y^{(2)}(t) \approx \frac{y^{(1)}(t) - y^{(1)}(t-h)}{h}$$

and then substitute into this the two approximations

$$y^{(1)}(t) \approx \frac{y(t) - y(t-h)}{h} \text{ and } y^{(1)}(t-h) \approx \frac{y(t-h) - y(t-2h)}{h}.$$

What formula do you get?

Answer: You should get an approximation we have already found.

13. Suppose we use the approximation $y^{(1)}(t) \approx \frac{y(t-h) - 2y(t-2h) + y(t-3h)}{h^2}$ to approximate the derivative one time step into the future of the three samples at $t - h$, $t - 2h$ and $t - 3h$. What is the error of this approximation?

Answer: The formula for approximating the second derivative using t, $t - h$ and $t - 2h$ is already O(h), so this can't be any better, so let's start with second-order Taylor series expansions:

$$y(t-h) = y(t) - y^{(1)}(t)h + \frac{1}{2}y^{(2)}(t)h^2 - \frac{1}{6}y^{(3)}(\tau_{-1})h^3$$

$$y(t-2h) = y(t) - 2y^{(1)}(t)h + 2y^{(2)}(t)h^2 - \frac{4}{3}y^{(3)}(\tau_{-2})h^3$$

$$y(t-3h) = y(t) - 3y^{(1)}(t)h + \frac{9}{2}y^{(2)}(t)h^2 - \frac{9}{2}y^{(3)}(\tau_{-3})h^3$$

Taking an appropriate linear combination of these, dividing by h^2 and isolating the 2nd derivative, we have

$$y^{(2)}(t) = \frac{y(t-h) - 2y(t-2h) + y(t-3h)}{h^2} - \left(-\frac{1}{6}y^{(3)}(\tau_{-1})h^3 + \frac{8}{3}y^{(3)}(\tau_{-2})h^3 - \frac{9}{2}y^{(3)}(\tau_{-3})\right)h$$

The second expression is not a convex combination (with all coefficients greater than or equal to zero), so the best we can say is:

$$y^{(2)}(t) = \frac{y(t-h) - 2y(t-2h) + y(t-3h)}{h^2} + 2y^{(3)}(t)h + O(h^2)$$

Thus, the error is $2y^{(3)}(t)h + O(h^2)$, which is O(h) and not O(h^2). Note that the error coefficient is twice as large as the error was for the approximation $y^{(1)}(t) \approx \frac{y(t) - 2y(t-h) + y(t-2h)}{h^2}$.

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